## PROOF OF THE DIVERGENCE THEOREM AND STOKES' THEOREM

In this section we give proofs of the Divergence Theorem and Stokes' Theorem using the definitions in Cartesian coordinates.

## Proof of the Divergence Theorem

Let $\vec{F}$ be a smooth vector field defined on a solid region $V$ with boundary surface $A$ oriented outward. We wish to show that

$$
\int_{A} \vec{F} \cdot d \vec{A}=\int_{V} d i v \vec{F} d V
$$

For the Divergence Theorem, we use the same approach as we used for Green's Theorem; first prove the theorem for rectangular regions, then use the change of variables formula to prove it for regions parameterized by rectangular regions, and finally paste such regions together to form general regions.

## Proof for Rectangular Solids with Sides Parallel to the Axes

Consider a smooth vector field $\vec{F}$ defined on the rectangular solid $V$ : $a \leq x \leq b, c \leq y \leq d$, $e \leq z \leq f$. (See Figure M.50). We start by computing the flux of $\vec{F}$ through the two faces of $V$ perpendicular to the $x$-axis, $A_{1}$ and $A_{2}$, both oriented outward:

$$
\begin{aligned}
\int_{A_{1}} \vec{F} \cdot d \vec{A}+\int_{A_{2}} \vec{F} \cdot d \vec{A} & =-\int_{e}^{f} \int_{c}^{d} F_{1}(a, y, z) d y d z+\int_{e}^{f} \int_{c}^{d} F_{1}(b, y, z) d y d z \\
& =\int_{e}^{f} \int_{c}^{d}\left(F_{1}(b, y, z)-F_{1}(a, y, z)\right) d y d z
\end{aligned}
$$

By the Fundamental Theorem of Calculus,

$$
F_{1}(b, y, z)-F_{1}(a, y, z)=\int_{a}^{b} \frac{\partial F_{1}}{\partial x} d x
$$

so

$$
\int_{A_{1}} \vec{F} \cdot d \vec{A}+\int_{A_{2}} \vec{F} \cdot d \vec{A}=\int_{e}^{f} \int_{c}^{d} \int_{a}^{b} \frac{\partial F_{1}}{\partial x} d x d y d z=\int_{V} \frac{\partial F_{1}}{\partial x} d V
$$

By a similar argument, we can show

$$
\int_{A_{3}} \vec{F} \cdot d \vec{A}+\int_{A_{4}} \vec{F} \cdot d \vec{A}=\int_{V} \frac{\partial F_{2}}{\partial y} d V \quad \text { and } \quad \int_{A_{5}} \vec{F} \cdot d \vec{A}+\int_{A_{6}} \vec{F} \cdot d \vec{A}=\int_{V} \frac{\partial F_{3}}{\partial z} d V
$$

Adding these, we get

$$
\int_{A} \vec{F} \cdot d \vec{A}=\int_{V}\left(\frac{\partial F_{1}}{\partial x}+\frac{\partial F_{2}}{\partial y}+\frac{\partial F_{3}}{\partial z}\right) d V=\int_{V} \operatorname{div} \vec{F} d V
$$

This is the Divergence Theorem for the region $V$.


## Proof for Regions Parameterized by Rectangular Solids

Now suppose we have a smooth change of coordinates

$$
x=x(s, t, u), \quad y=y(s, t, u), \quad z=z(s, t, u)
$$

Consider a curved solid $V$ in $x y z$-space corresponding to a rectangular solid $W$ in $s t u$-space. See Figure M.51. We suppose that the change of coordinates is one-to-one on the interior of $W$, and that its Jacobian determinant is positive on $W$. We prove the Divergence Theorem for $V$ using the Divergence Theorem for $W$.

Let $A$ be the boundary of $V$. To prove the Divergence Theorem for $V$, we must show that

$$
\int_{A} \vec{F} \cdot d \vec{A}=\int_{V} \operatorname{div} \vec{F} d V
$$

First we express the flux through $A$ as a flux integral in $s t u$-space over $S$, the boundary of the rectangular region $W$. In vector notation the change of coordinates is

$$
\vec{r}=\vec{r}(s, t, u)=x(s, t, u) \vec{i}+y(s, t, u) \vec{j}+z(s, t, u) \vec{k} .
$$

The face $A_{1}$ of $V$ is parameterized by

$$
\vec{r}=\vec{r}(a, t, u), \quad c \leq t \leq d, \quad e \leq u \leq f
$$

so on this face

$$
d \vec{A}= \pm \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u} d t d u
$$

In fact, in order to make $d \vec{A}$ point outward, we must choose the negative sign. (Problem 3 on page 73 shows how this follows from the fact that the Jacobian determinant is positive.) Thus, if $S_{1}$ is the face $s=a$ of $W$,

$$
\int_{A_{1}} \vec{F} \cdot d \vec{A}=-\int_{S_{1}} \vec{F} \cdot \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u} d t d u
$$

The outward pointing area element on $S_{1}$ is $d \vec{S}=-\vec{i} d t d u$. Therefore, if we choose a vector field $\vec{G}$ on $s t u$-space whose component in the $s$-direction is

$$
G_{1}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u}
$$

we have

$$
\int_{A_{1}} \vec{F} \cdot d \vec{A}=\int_{S_{1}} \vec{G} \cdot d \vec{S}
$$

Similarly, if we define the $t$ and $u$ components of $\vec{G}$ by

$$
G_{2}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial s} \quad \text { and } \quad G_{3}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}
$$

then

$$
\int_{A_{i}} \vec{F} \cdot d \vec{A}=\int_{S_{i}} \vec{G} \cdot d \vec{S}, \quad i=2, \ldots, 6
$$

(See Problem 4.) Adding the integrals for all the faces, we find that

$$
\int_{A} \vec{F} \cdot d \vec{A}=\int_{S} \vec{G} \cdot d \vec{S}
$$

Since we have already proved the Divergence Theorem for the rectangular region $W$, we have

$$
\int_{S} \vec{G} \cdot d \vec{S}=\int_{W} \operatorname{div} \vec{G} d W
$$

where

$$
\operatorname{div} \vec{G}=\frac{\partial G_{1}}{\partial s}+\frac{\partial G_{2}}{\partial t}+\frac{\partial G_{3}}{\partial u}
$$

Problems 5 and 6 on page 73 show that

$$
\frac{\partial G_{1}}{\partial s}+\frac{\partial G_{2}}{\partial t}+\frac{\partial G_{3}}{\partial u}=\left|\frac{\partial(x, y, z)}{\partial(s, t, u)}\right|\left(\frac{\partial F_{1}}{\partial x}+\frac{\partial F_{2}}{\partial y}+\frac{\partial F_{3}}{\partial z}\right)
$$

So, by the three-variable change of variables formula on page 61,

$$
\begin{aligned}
\int_{V} \operatorname{div} \vec{F} d V & =\int_{V}\left(\frac{\partial F_{1}}{\partial x}+\frac{\partial F_{2}}{\partial y}+\frac{\partial F_{3}}{\partial z}\right) d x d y d z \\
& =\int_{W}\left(\frac{\partial F_{1}}{\partial x}+\frac{\partial F_{2}}{\partial y}+\frac{\partial F_{3}}{\partial z}\right)\left|\frac{\partial(x, y, z)}{\partial(s, t, u)}\right| d s d t d u \\
& =\int_{W}\left(\frac{\partial G_{1}}{\partial s}+\frac{\partial G_{2}}{\partial t}+\frac{\partial G_{3}}{\partial u}\right) d s d t d u \\
& =\int_{W} \operatorname{div} \vec{G} d W
\end{aligned}
$$

In summary, we have shown that

$$
\int_{A} \vec{F} \cdot d \vec{A}=\int_{S} \vec{G} \cdot d \vec{S}
$$

and

$$
\int_{V} \operatorname{div} \vec{F} d V=\int_{W} \operatorname{div} \vec{G} d W
$$

By the Divergence Theorem for rectangular solids, the right-hand sides of these equations are equal, so the left-hand sides are equal also. This proves the Divergence Theorem for the curved region $V$.

## Pasting Regions Together

As in the proof of Green's Theorem, we prove the Divergence Theorem for more general regions by pasting smaller regions together along common faces. Suppose the solid region $V$ is formed by pasting together solids $V_{1}$ and $V_{2}$ along a common face, as in Figure M. 52 .

The surface $A$ which bounds $V$ is formed by joining the surfaces $A_{1}$ and $A_{2}$ which bound $V_{1}$ and $V_{2}$, and then deleting the common face. The outward flux integral of a vector field $\vec{F}$ through $A_{1}$ includes the integral across the common face, and the outward flux integral of $\vec{F}$ through $A_{2}$


Figure M.52: Region $V$ formed by pasting together $V_{1}$ and $V_{2}$
includes the integral over the same face, but oriented in the opposite direction. Thus, when we add the integrals together, the contributions from the common face cancel, and we get the flux integral through $A$. Thus we have

$$
\int_{A} \vec{F} \cdot d \vec{A}=\int_{A_{1}} \vec{F} \cdot d \vec{A}+\int_{A_{2}} \vec{F} \cdot d \vec{A}
$$

But we also have

$$
\int_{V} \operatorname{div} \vec{F} d V=\int_{V_{1}} \operatorname{div} \vec{F} d V+\int_{V_{2}} \operatorname{div} \vec{F} d V
$$

So the Divergence Theorem for $V$ follows from the Divergence Theorem for $V_{1}$ and $V_{2}$. Hence we have proved the Divergence Theorem for any region formed by pasting together regions that can be smoothly parameterized by rectangular solids.

Example 1 Let $V$ be a spherical ball of radius 2, centered at the origin, with a concentric ball of radius 1 removed. Using spherical coordinates, show that the proof of the Divergence Theorem we have given applies to $V$.

Solution We cut $V$ into two hollowed hemispheres like the one shown in Figure M.53, $W$. In spherical coordinates, $W$ is the rectangle $1 \leq \rho \leq 2,0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi$. Each face of this rectangle becomes part of the boundary of $W$. The faces $\rho=1$ and $\rho=2$ become the inner and outer hemispherical surfaces that form part of the boundary of $W$. The faces $\theta=0$ and $\theta=\pi$ become the two halves of the flat part of the boundary of $W$. The faces $\phi=0$ and $\phi=\pi$ become line segments along the $z$-axis. We can form $V$ by pasting together two solid regions like $W$ along the flat surfaces where $\theta=$ constant.



Figure M.53: The hollow hemisphere $W$ and the corresponding rectangular region in $\rho \theta \phi$-space

## Proof of Stokes' Theorem

Consider an oriented surface $A$, bounded by the curve $B$. We want to prove Stokes' Theorem:

$$
\int_{A} \operatorname{curl} \vec{F} \cdot d \vec{A}=\int_{B} \vec{F} \cdot d \vec{r}
$$

We suppose that $A$ has a smooth parameterization $\vec{r}=\vec{r}(s, t)$, so that $A$ corresponds to a region $R$ in the st-plane, and $B$ corresponds to the boundary $C$ of $R$. See Figure M.54. We prove Stokes' Theorem for the surface $A$ and a continuously differentiable vector field $\vec{F}$ by expressing the integrals on both sides of the theorem in terms of $s$ and $t$, and using Green's Theorem in the st-plane.

First, we convert the line integral $\int_{B} \vec{F} \cdot d \vec{r}$ into a line integral around $C$ :

$$
\int_{B} \vec{F} \cdot d \vec{r}=\int_{C} \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} d s+\vec{F} \cdot \frac{\partial \vec{r}}{\partial t} d t
$$

So if we define a 2-dimensional vector field $\vec{G}=\left(G_{1}, G_{2}\right)$ on the $s t$-plane by

$$
G_{1}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \quad \text { and } \quad G_{2}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial t}
$$

then

$$
\int_{B} \vec{F} \cdot d \vec{r}=\int_{C} \vec{G} \cdot d \vec{s}
$$

using $\vec{s}$ to denote the position vector of a point in the $s t$-plane.
What about the flux integral $\int_{A} \operatorname{curl} \vec{F} \cdot d \vec{A}$ that occurs on the other side of Stokes' Theorem? In terms of the parameterization,

$$
\int_{A} \operatorname{curl} \vec{F} \cdot d \vec{A}=\int_{R} \operatorname{curl} \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} d s d t
$$

In Problem 7 on page 74 we show that

$$
\operatorname{curl} \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}=\frac{\partial G_{2}}{\partial s}-\frac{\partial G_{1}}{\partial t}
$$

Hence

$$
\int_{A} \operatorname{curl} \vec{F} \cdot d \vec{A}=\int_{R}\left(\frac{\partial G_{2}}{\partial s}-\frac{\partial G_{1}}{\partial t}\right) d s d t
$$

We have already seen that

$$
\int_{B} \vec{F} \cdot d \vec{r}=\int_{C} \vec{G} \cdot d \vec{s}
$$

By Green's Theorem, the right-hand sides of the last two equations are equal. Hence the left-hand sides are equal as well, which is what we had to prove for Stokes' Theorem.


Figure M.54: A region $R$ in the st-plane and the corresponding surface $A$ in $x y z$-space; the curve $C$ corresponds to the boundary of $B$

## Problems for Section M

1. Let $W$ be a solid circular cylinder along the $z$-axis, with a smaller concentric cylinder removed. Parameterize $W$ by a rectangular solid in $r \theta z$-space, where $r, \theta$, and $z$ are cylindrical coordinates.
2. In this section we proved the Divergence Theorem using the coordinate definition of divergence. Now we use the Divergence Theorem to show that the coordinate definition is the same as the geometric definition. Suppose $\vec{F}$ is smooth in a neighborhood of $\left(x_{0}, y_{0}, z_{0}\right)$, and let $U_{R}$ be the ball of radius $R$ with center $\left(x_{0}, y_{0}, z_{0}\right)$. Let $m_{R}$ be the minimum value of $\operatorname{div} \vec{F}$ on $U_{R}$ and let $M_{R}$ be the maximum value.
(a) Let $S_{R}$ be the sphere bounding $U_{R}$. Show that

$$
m_{R} \leq \frac{\int_{S_{R}} \vec{F} \cdot d \vec{A}}{\text { Volume of } U_{R}} \leq M_{R}
$$

(b) Explain why we can conclude that

$$
\lim _{R \rightarrow 0} \frac{\int_{S_{R}} \vec{F} \cdot d \vec{A}}{\text { Volume of } U_{R}}=\operatorname{div} \vec{F}\left(x_{0}, y_{0}, z_{0}\right)
$$

(c) Explain why the statement in part (b) remains true if we replace $U_{R}$ with a cube of side $R$, centered at $\left(x_{0}, y_{0}, z_{0}\right)$.

Problems 3-6 fill in the details of the proof of the Divergence Theorem.
3. Figure M. 51 on page 69 shows the solid region $V$ in $x y z$ space parameterized by a rectangular solid $W$ in stuspace using the continuously differentiable change of coordinates

$$
\vec{r}=\vec{r}(s, t, u), \quad a \leq s \leq b, c \leq t \leq d, e \leq u \leq f
$$

Suppose that $\frac{\partial \vec{r}}{\partial s} \cdot\left(\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u}\right)$ is positive.
(a) Let $A_{1}$ be the face of $V$ corresponding to the face $s=a$ of $W$. Show that $\frac{\partial \vec{r}}{\partial s}$, if it is not zero, points into $W$.
(b) Show that $-\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u}$ is an outward pointing normal on $A_{1}$.
(c) Find an outward pointing normal on $A_{2}$, the face of $V$ where $s=b$.
4. Show that for the other five faces of the solid $V$ in the proof of the Divergence Theorem (see page 70):

$$
\int_{A_{i}} \vec{F} \cdot d \vec{A}=\int_{S_{i}} \vec{G} \cdot d \vec{S}, \quad i=2,3,4,5,6 .
$$

5. Suppose that $\vec{F}$ is a continuously differentiable vector field and that $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors. In this problem we prove the formula

$$
\begin{aligned}
& \operatorname{grad}(\vec{F} \cdot \vec{b} \times \vec{c}) \cdot \vec{a}+\operatorname{grad}(\vec{F} \cdot \vec{c} \times \vec{a}) \cdot \vec{b} \\
& +\operatorname{grad}(\vec{F} \cdot \vec{a} \times \vec{b}) \cdot \vec{c}=(\vec{a} \cdot \vec{b} \times \vec{c}) \operatorname{div} \vec{F} .
\end{aligned}
$$

(a) Interpreting the divergence as flux density, explain why the formula makes sense. [Hint: Consider the flux out of a small parallelepiped with edges parallel to $\vec{a}, \vec{b}, \vec{c}$.]
(b) Say how many terms there are in the expansion of the left-hand side of the formula in Cartesian coordinates, without actually doing the expansion.
(c) Write down all the terms on the left-hand side that contain $\partial F_{1} / \partial x$. Show that these terms add up to $\vec{a} \cdot \vec{b} \times \vec{c} \frac{\partial F_{1}}{\partial x}$.
(d) Write down all the terms that contain $\partial F_{1} / \partial y$. Show that these add to zero.
(e) Explain how the expressions involving the other seven partial derivatives will work out, and how this verifies that the formula holds.
6. Let $\vec{F}$ be a smooth vector field in 3 -space, and let

$$
x=x(s, t, u), \quad y=y(s, t, u), \quad z=z(s, t, u)
$$

be a smooth change of variables, which we will write in vector form as
$\vec{r}=\vec{r}(s, t, u)=x(s, t, u) \vec{i}+y(s, t, u) \vec{j}+z(s, t, u) \vec{k}$.
Define a vector field $\vec{G}=\left(G_{1}, G_{2}, G_{3}\right)$ on $s t u$-space by

$$
\begin{aligned}
G_{1} & =\vec{F} \cdot \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u} \quad G_{2}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial s} \\
G_{3} & =\vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}
\end{aligned}
$$

(a) Show that

$$
\begin{aligned}
& \frac{\partial G_{1}}{\partial s}+\frac{\partial G_{2}}{\partial t}+\frac{\partial G_{3}}{\partial u}=\frac{\partial \vec{F}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u} \\
& +\frac{\partial \vec{F}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial s}+\frac{\partial \vec{F}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}
\end{aligned}
$$

(b) Let $\vec{r}_{0}=\vec{r}\left(s_{0}, t_{0}, u_{0}\right)$, and let

$$
\vec{a}=\frac{\partial \vec{r}}{\partial s}\left(\vec{r}_{0}\right), \quad \vec{b}=\frac{\partial \vec{r}}{\partial t}\left(\vec{r}_{0}\right), \quad \vec{c}=\frac{\partial \vec{r}}{\partial u}\left(\vec{r}_{0}\right) .
$$

Use the chain rule to show that

$$
\begin{aligned}
& \left.\left(\frac{\partial G_{1}}{\partial s}+\frac{\partial G_{2}}{\partial t}+\frac{\partial G_{3}}{\partial u}\right)\right|_{\vec{r}=\vec{r}_{0}}= \\
& \operatorname{grad}(\vec{F} \cdot \vec{b} \times \vec{c}) \cdot \vec{a}+\operatorname{grad}(\vec{F} \cdot \vec{c} \times \vec{a}) \cdot \vec{b} \\
& +\operatorname{grad}(\vec{F} \cdot \vec{a} \times \vec{b}) \cdot \vec{c}
\end{aligned}
$$

(c) Use Problem 5 to show that

$$
\begin{aligned}
& \frac{\partial G_{1}}{\partial s}+\frac{\partial G_{2}}{\partial t}+\frac{\partial G_{3}}{\partial u}= \\
& \left|\frac{\partial(x, y, z)}{\partial(s, t, u)}\right|\left(\frac{\partial F_{1}}{\partial x}+\frac{\partial F_{2}}{\partial y}+\frac{\partial F_{3}}{\partial z}\right) .
\end{aligned}
$$

7. This problem completes the proof of Stokes' Theorem. Let $\vec{F}$ be a smooth vector field in 3 -space, and let $S$ be a surface parameterized by $\vec{r}=\vec{r}(s, t)$. Let $\vec{r}_{0}=$ $\vec{r}\left(s_{0}, t_{0}\right)$ be a fixed point on $S$. We define a vector field in $s t$-space as on page 72 :

$$
G_{1}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \quad G_{2}=\vec{F} \cdot \frac{\partial \vec{r}}{\partial t} .
$$

(a) Let $\vec{a}=\frac{\partial \vec{r}}{\partial s}\left(\vec{r}_{0}\right), \quad \vec{b}=\frac{\partial \vec{r}}{\partial t}\left(\vec{r}_{0}\right)$. Show that

$$
\begin{aligned}
& \frac{\partial G_{1}}{\partial t}\left(\vec{r}_{0}\right)-\frac{\partial G_{2}}{\partial s}\left(\vec{r}_{0}\right)= \\
& \operatorname{grad}(\vec{F} \cdot \vec{a}) \cdot \vec{b}-\operatorname{grad}(\vec{F} \cdot \vec{b}) \cdot \vec{a} .
\end{aligned}
$$

(b) Use Problem 30 on page 961 of the textbook to show

$$
\operatorname{curl} \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}=\frac{\partial G_{2}}{\partial s}-\frac{\partial G_{1}}{\partial t}
$$

